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FUNCTIONALITY IN MATHEMATICAL INSTRUCTION IN SCHOOLS AND COLLEGES

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(Address delivered before the Mathematical Association of America. Toronto, December 30, 1921.)

1. *General Ideas.* The topic on which I am to speak to you was emphasized by the National Committee on Mathematical Requirements of this Association in its first report entitled "The Reorganization of the First Courses in Secondary School Mathematics," which was published as Secondary School Circular No. 5 by the U. S. Bureau of Education. In that report, the Committee says:

> General Ideas.—The one great idea which is sufficient in scope to unify the course is that of the functional relation. The concept of a variable and of the dependence of one variable upon another is of fundamental importance for everyone. It is true that the general and abstract form of these concepts can become significant to the pupil only as a result of very considerable mathematical experience and training. There is nothing in either concept, however, which prevents the presentation of specific concrete examples and illustrations of dependence even in the early parts of the course. Means to this end will be found in connection with the tabulation of data, and the study of the formula and the graph and of their uses.
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> The primary and underlying principle of the course, particularly in connection with algebra and trigonometry, should be the notion of relationship between variables, including the methods of determining and expressing such relationship. The notion of relationship is fundamental both in algebra and in geometry. The teacher should have it constantly in mind, and the pupil's advancement should be consciously directed along the lines which will present first one and then another of the details upon which finally the formation of the general concept of functionality depends.

Although this statement contains in brief the substance of the whole matter, it was felt that some elaboration of the ideas was necessary to convey adequately to teachers its full import. Accordingly, the Committee requested me to prepare a more detailed statement; and this statement, after some revision, was adopted by the Committee, and was issued under the title "The Function Concept in Secondary School Mathematics" by the U. S. Bureau of Education as Secondary School Circular No. 8, June, 1921. I am assuming that many, if not all of you, have seen these two reports, and I shall not repeat them unduly.

These, as well as other reports of the Committee, may be obtained through the Chairman of the Committee or directly from Washington; they will be contained also in the full report of the Committee, which is soon to be issued in final form.

My address today would have been incomplete without at least some mention of these reports. On the other hand, these reports refer almost exclusively to the early work of the first two years of Secondary Schools. The principles announced in them are of far wider application, however, and I desire to present to you today what I regard as the very vital application of these principles not alone for mathematics, but even more for education as a whole; and not alone in secondary schools, but even more in colleges and universities. I shall point out in detail the opportunity and the need for emphasis on the functional elements in mathematical teaching in schools of every grade; and I shall try to show that this type of mathematics is of direct value, both to the individual for his selfish or personal good, and also to society and to the state, for the advancement of scientific knowledge, for the social and economic problems of the community that involve quantitative relations, and for the development of more intelligent citizenship. These phases of mathematical instruction have been too long ignored not only by professors of education who may be blamelessly ignorant of their very existence, but also by teachers and professors of mathematics, in whom ignorance is inexcusable. To continue to ignore what is probably the greatest asset of mathematics as an element in public education is not merely a demonstration of inefficiency; it is a crime against mathematics and against the cause of education in general.

2. *The Function Idea.* To such an audience as this it is unnecessary to explain what is meant by the function idea. Indeed, the danger would seem to me to be that your great familiarity with this notion may breed contempt of its elementary phases, and contempt of those who innocently know nothing about it. Denuded of our traditional phraseology, the idea of functionality is nothing else than the notion of relationships between quantities, and the manner in which changes in one of two related quantities produces—or is accompanied by—changes in the other. To you it will seem that the existence of such relationships is so self-evident, and that the effects of changes in one

quantity upon another are so easy to trace, that we might assume such ideas as part of the mental equipment of all human beings. This is far from true. The acquisition of such ideas is a very slow process. It must be begun early in a very simple manner; it must be presented first only by individual instances of a simple and numerical character; it must be fixed in the mind by repetition after repetition, and by instance after instance, until thinking in such terms becomes habitual with the individual. Only in this way can an individual acquire what has been called the "habit of functional thinking."

That relationships between quantities, and the effects upon one quantity of changes in a related quantity, enter into all human thinking about quantities, including a great part of the daily activities of ordinary life, will be admitted by all. The danger is that those already competent in such thinking will not recognize any need for training in it; and in particular that teachers of mathematics will overlook this obvious element in mathematical teaching because it seems to them self-explanatory. There will be many who think habitually about quantitative relations who will be unconscious of the mathematical foundation for the very thinking that they do.

The point I am striving to make is most clearly illustrated by a comparison with the ordinary use of English in speaking and in writing. Beneath the conscious habit of swift use of correct English there lies a subconscious knowledge of grammatical form acquired either by conscious study or by the more painful path of rough experience. Without this substratum of grammatical form, correct use of English is not only difficult—it is unthinkable. To acquire a correct and ready command of English, one may follow either the ready path of conscious education, or the thornier one of experience by trial and error. However acquired, the goal toward which we strive is the submergence into subconsciousness of the slow rules of grammar, and the formation of habits which alone render possible that swiftness of application which is indispensable in actual speech.

Similarly, thinking about quantities cannot be conducted with the necessary facility and speed until all conscious thought of algebraic rules has sunk into the subconscious portions of our minds, and until correct habits of thought concerning relationships between quantities have become so ingrained that we are totally unconscious of the use of algebra, just as we are totally unconscious of the use of grammar when we speak. To assert that correct thinking about quantitative relationships is impossible without a previous study of algebra is just as false as to assert that correct speech is impossible without a previous conscious study of grammar. To assert that any course in algebra—for example, a course filled with formal definitions and practice in the manipulation of symbols to the exclusion of thought processes about quantitative relationships—to assert that any such course will furnish the requisite background for correct thinking about quantitative relationships that occur in actual life, is just as foolish as to suppose that any course in grammar—for example, a course wholly given over to formal definitions of parts of speech, and to declensions and conjugations—would furnish an adequate background for correct usage of English.

It is not definitions that we need. For my part, I would as leave have a student utterly unaware of the existence of the word "function." What we do want is that the student shall be vividly aware of the notion itself, of the existence of actual functions, of the meaning of a variety of relationships between quantities whose meaning is clear to him, and of the manner in which changes in one quantity affect the values of another, associated, quantity. This can be attained if we attempt it; it cannot be attained by keeping our courses solely to formalities of definitions and manipulations of symbols; it cannot be attained by the drawing of a few graphs in a perfunctory manner; it cannot be attained by devoting three recitations—or three months—to learning set definitions; it cannot be learned by any means readily tested by formal examinations; it can be learned if we teachers of mathematics solemnly resolve to call attention to every case in which quantities are related, and if we require attention and thought upon the manner in which changes in one quantity are associated with changes in another.

It may be asked how and when opportunity for such discussions will arise. The answer is that such an opportunity presents itself whenever we are dealing with quantities that are in any way related to other quantities. Lack of such opportunity means either that we are not dealing with quantities, or that we are dealing with quantities only one at a time, and that those we use are not related to or dependent upon any others.

Now quantities that have any meaning usually are related to other quantities, and their values are affected by the values of those others. How may we manage, indeed, to deal with quantities at all without dealing with related quantities? The only way to escape relationships between quantities is to deal with quantities so abstract that the relationships are lost to sight. Does it not occur to you that such abstraction is rather common in our traditional courses?

Functional relations—that is relations between quantities—will occur on every page of every book on mathematics unless we suppress them. We have been suppressing them. It is our business to emphasize them, not during one week nor one month, nor one year, but precisely whenever they occur. Since mathematics deals almost exclusively with relations between quantities, you may be fairly sure that it is something other than mathematics which is being emphasized if such relationships do not appear. And this is true. In algebra, the shorthand has supplanted the real mathematics to an alarming degree. In geometry, formal logic has all but banished mathematical thinking. Shorthand in algebra is extremely desirable—in order to express relationships between quantities; but to spend all the time on the shorthand—the symbolism—without any realization of the relationships for which the shorthand was to be the handmaiden, is to spread a feast without any guests. Logic is our supreme instrument for the discovery and confirmation of geometric and other relationships; but I have seen courses in geometry so crammed with formal logic that geometric relationships utterly failed to appear, just as a child may gorge itself with food that would furnish needed energy until all action for which the food would be good is utterly estopped.

It is such courses in algebra and in geometry that have furnished just cause for educational criticism. Algebra that has turned into shorthand, and shorthand that is never used. Geometry that is turned into formal logic, to the abandonment of geometric relationships. Such algebra, such geometry, these are no longer branches of mathematics if quantitative relations are not studied.

3. *Review of Algebra.* The criterion as to whether a given piece of work is algebraic is not the number of symbols it contains. It may or it may not contain symbols. The real criterion

is whether there is in it any question of relationship between numerical quantities. The false standard by which algebra is often judged, by the signs and symbols and the detached letters, is a criterion of shorthand only. Algebra can be written without shorthand. Shorthand can be written without algebraic content. A course in algebra should be a course in relationships between numerical quantities, with as much shorthand as may be convenient. The courses criticized are often little more than courses in shorthand, with as little algebra—*i. e.*, as little quantitative relationship, as can be managed.

In the report on the function concept mentioned above, I have tried to bring out in detail the variety of quantitative relationships that exist in our courses in algebra, if only we do not suppress them. Graphs, to be sure; but graphs of quantitative situations within the grasp of the child, not graphs of formalized equations. I have not emphasized graphs at all, for the danger is that the hasty teacher may see nothing else. Every instance of the use of letters for quantities is only to express a relationship; otherwise it is really quite as useless as some educators believe it to be. Why should we write for simple interest $i = r \cdot p \cdot t$ unless we are at least to contemplate the relationship, and to note how the interest is affected by changes in the rate, principal, and time? More complex forms of relationships occur later in algebra, through formulas of science introduced as applications, and through formulas of purely mathematical origin. Thus I have emphasized such formulas as the laws of falling bodies, the law for compound interest, the law for the square of a sum. In all such cases, real appreciation is probably lacking if or until the relationships between the quantities involved is clearly thought out and understood. The contrast between the total interest at simple interest and that at compound interest is a striking case. I have detailed all this in my report more fully than I can do here.

May I emphasize the growing importance of tables of numerical values? Any such table, in the nature of things, tells many values of one of two related quantities when the other is given. It is strictly functional in character. The need for interpolation in such a table also forces functional thinking of a high order, if it is intelligently done.

Finally, the very existence of what we call practical problems depends upon the relationships between the quantities mentioned in the problem. If only one concept were mentioned, the problem must be extremely artificial or extremely simple. The study of the relations between the quantities mentioned in the problem will ordinarily lead to a solution. Failure to study these relations is the most usual source of trouble in such problems.

4. *Review of Geometry.* The case in geometry is no less striking. Here the fundamentals of the subject are the relationships that exist between geometric quantities, as opposed to the relationships between numerical quantities which constitute algebra. The relations that exist between the parts of a triangle have been developed most completely, and we recognize their functional character when we use the so-called trigonometric functions. But the parts of all geometric figures are inter-related. Geometry should make these relations stand forth. I well remember that my own first course in geometry did nothing of the sort. To watch the shape of a parallelogram change as the angles change while the sides remain fixed; to think out the resulting changes in the lengths of the diagonals; all this is real geometry. To consider the arcs that measure an angle as the vertex of the angle moves from the center of the circle out toward the circumference and then beyond the circumference, is to acquire real insight into these geometric forms.

I do not decry in the slightest degree the necessity of formal logic in geometry. It is necessary. But it is not the subject-matter of the geometry. The subject-matter consists in those relationships between geometric forms which the logic serves to demonstrate.

Changes produced in figures by the variation of one or more of its parts are of the utmost consequence not only toward a real mastery of geometry, but also for the applications. Thus a knowledge of which parts of a framework are necessary to make it rigid, and which must break if it is to change form, are of the utmost consequence in all building operations. Ability to think such geometric relationships quickly and accurately is a part of the reasonable mental equipment of all grown men and of most women. I do not think that it is taught adequately in our courses in geometry.

Motion, and the properties of moving figures, occur in the lives of all people. To think accurately about machinery, about plans, about patterns, about motions, geometric habits of functional thinking are quite necessary. A study of geometry, to be thorough and to make for efficiency in life, demands habits of functional thinking regarding geometric forms. Without it, geometry sinks to a level of isolated figures, and is often little more than a quickly forgotten exercise in logistics.

5. *Algebra for Girls. Other Special Classes.* A question that I have often heard asked, and one that still amazes me, is concerning the usefulness of algebra for girls. Perhaps this question is inspired by that purely formalistic view of algebra that takes no account of the formation of the habit of functional thinking, *i. e.,* the habit of thinking about relations between quantities. For every woman who does anything, whether it be a man-like task outside the home, or those tasks of housewifery that are often so misprized, deals with quantities and with relations between quantities every day of every week. The question as to whether twice as much gas burned under a kettle will make potatoes cook twice as fast is of the same order of difficulty as the question as to whether compound interest on money for twice as long is twice as much. The question as to the comparative contents of two cans of tomatoes of the same shape is actually the same question as the question as to the relative volumes of similar cylinders. If one is twice as high as the other, will it hold just twice as much? Will twice as much ice make the icebox twice as cold? What is meant by twice as cold? The reading of a gasmeter is a functional question of no mean order. Will twice as much sugar make a thing twice as sweet? Will doubling the amount of soap wash the clothes twice as clean? Will doubling the amount of yeast make the loaves twice as large, or raise the bread twice as fast? What is the effect on cake of increasing the amount of butter? Are oranges twice as thick through twice as heavy? If the skins are twice as thick on the large oranges, did we miscalculate the relative values of the oranges? Does it pay to buy big potatoes as compared with small ones? Will a mail order package twice as heavy cost twice as much postage? If sent by express, would the expressage be doubled? When does it become advisable to ship by freight? Will a kitchen window twice as large let in twice as much light?

Will twice as many guests double the expense of a party: hence is it just as well to have the same number of guests at two different times as to have all at one time? Will twice the amount of food at each of half as many meals sustain life as well? Does one feed a man who works more than one who does not? How much more? If he works twice as long, should he have twice as much food? And there are those finer questions of balancing rations for children, for invalids, for old people; questions of calories and proteins and carbohydrates for the real expert; questions of bacterial growth in diseases, and in sour milk, and—sometimes—in beverages; questions of food for a mother and food for a child, content of butter fat and protein in milk and in skimmed milk, acids, sugars, starches. Quantities—quantities—relations between quantities.

Perhaps many a woman hesitates to comprehend or even to study such questions. "There are too many quantities." "I can never figure things out." "I have no head for figures." "That is too complicated for me." But all these things are well within the range of experience of most women. Why are they too hard? Why cannot women understand and master such things? Is it perhaps that they have not formed habits of quantitative thinking? Is it perhaps that some all-too-wise person has told them that algebra is of no use to girls? Is it perhaps that the teacher of algebra was too busy with the shorthand of algebraic symbols to spend time over mere questioning about relationships between quantities?

Some girls need no experience in dealing with quantities, and no habit of accurate thinking about quantitative relations. Some girls will never have to do with quantities or with relations between quantities. Who are these girls? They are those who toil not, neither do they spin. The idle have no need for accurate habits of thought. The parasite on society needs no knowledge of relationships between quantities. But all other women meet with quantities and with relationships between them. Training for parasitism—and there is woefully much of just that—need include nothing about quantities. Training for usefulness—which is doubtless far drearier—should not neglect training in habits of quantitative thinking, training of such character that finally accurate thinking is possible speedily and without consciousness of the sub-conscious basis for that accuracy, just as speech should be accurate and rapid without thought of the sub-conscious basis for language. Those who plan curricula for girls should not overlook these things. Nor should those who teach mathematics to girls. More habits of quantitative thinking—and less shorthand. They will have need for all the knowledge they can get about quantitative relationships; they will have little need for manipulation of symbols and for intricate shorthand.

6. *College Mathematics.* Boys, too, have occasion to deal with quantities and to think accurately about relationships between quantities. I suppose that most people think that men deal with quantities more than women do. It was for that reason that I discussed the quantities women meet with somewhat in detail. If men meet more, they surely meet enough.

I have heard men say that they could never comprehend house plans, or plans for such complex structures as modern sewage systems. I know that many men cannot comprehend life insurance; hence they trust in that broken reed—the life insurance agent—blind leading the blind. We have been told rather frequently by those who have axes to grind that we—the common people—cannot understand such complex questions as railroad rates; and a few have asserted that only those who hold "a divine right" can comprehend, administer, and reap what profit they will, from these and other public corporations. Here is a very welter of quantities: compound interest, insurance, probabilities, depreciation.

I am not now speaking of such technically trained men as engineers, chemists, and other specialists. These are perfectly aware of their needs, and they are going to find the opportunity of meeting them, even when departments of mathematics are not framing their courses so as to meet these needs. They will be served. What we fail to give they will—and they should—seek elsewhere; and they will find . . . Remember descriptive geometry—a beautiful topic now all but lost to mathematics.

I choose to speak rather of that broader class who are being trained in colleges to go out into the world in dozens of different callings not so specifically technical as is engineering. The business man, in any of a hundred businesses, more especially the banker, but also good farmers, good lawyers, good storekeepers, good contractors, good manufacturers, good public service corporation men, good insurance men, all will find in their private interests not less but more problems of a quantitative nature, more relationships between quantities than those I have mentioned in detail for the housewife. While any one of them would probably be amazed at the variety of quantitative situations which the housewife has to meet, I doubt not that any one of them would stoutly contend that his own problems are still more varied and complex. Not to tarry over the pettier ones, let me mention one quite complex problem which is met by every class just mentioned: the problem of depreciation. On the farm, if it is a good farm, tools deteriorate, and they should be valued; in a factory, machinery deteriorates, buildings deteriorate, men deteriorate; in a public service corporation, all installation deteriorates, all is subject to public valuation; the part of the banker, the lawyer, the legislator, in all of this is evident upon very slight examination. We have met such problems very badly. Better business, higher efficiency, better service to the public, and better returns to the investor, all wait upon more general comprehension of the reality of these problems and of the possibility of solving them. Waste and inefficiency, lavish buying and prodigal carelessness of the future, these go hand in hand with guessing at quantities and at the effects of some on others. Careful and true thinking on these quantitative matters will save business from its precarious position, will save untold millions of public money and of private graft, will reassure a people now harassed and worried by private and public questions that they profess not to understand because of their quantitative complexities.

I say these things. Yet do our collegiate courses tend to train our students to meet all such quantitative situations when they do meet them? To lead the less fortunate on private and public affairs that involve quantities? I doubt it. Nor is the remedy simply in the substitution of a course in the mathematics of business for a course in trigonometry. In my opinion, both are too special in their scope and too narrowly technical and manipulative in their details. We should not seek to train the classes that face us wholly for surveying, nor wholly for insurance; nor wholly for accounting. It seems to me that we should, at least in a general course not designed for engineers nor for schools of business, give the great body of students as broad an idea of the variety of situations with which mathematics deals as we can possibly give. For them, the tangent law of trigonometry is as futile as a chapter from Bowditch's *Navigation.* But the familiarity with simple ideas such as the possibility of indirect measurement associated with right triangle solutions and tables of values, will be as likely to inspire him and to affect his life as would, for example, a knowledge of the Magna Charta, or a Description of the Glacial Period, or the Mendelian Law. Some knowledge of insurance there should be, but not a technical training in it outside of schools of business. To see that a net premium can be computed may really awaken intellectual interests undreamed of, and may seriously affect a career; at least, it is comparable with a knowledge of the Structure of the Inner Ear, or the Neanderthal Man, or Grimm's Law. But to carry the detail too far—say to actual computations with commutation columns—is to exclude the possibility of enrichment of the course by other ideas equally inviting. Thus probability, as applied in statistics, or in animal breeding, is of just as general importance and interest. But I am not intending to outline a course, or to favor any existing treatment: frankly I do not know a book that does all these thhings and also all the other things that I would class with them.

We should teach graphical processes, of course, to train for problems of life that involve quantitative relations. We should treat also the equations that are associated with these graphs; *i. e.,* we should teach analytic geometry. But again, we should have an eye to limitation on detail and perhaps another eye to the forceful presentation of the functional character of what we do present. Thus I gravely doubt whether the ordinary college man not intending to specialize in mathematics or engineering will profit either now or later from a treatment of the Normal Form of the Straight Line Equation. But I know that any one who is to meet and wrestle with quantitative relationships will profit decidedly from an appreciation of the nature of linear relationships, and the distinction between these and non--linear ones, even if he gets no further than to realize that non-linear relations are apt to occur. Apparently, many scientists and engineers have that yet to grasp effectively. On the other hand, are even our most detailed courses on analytics, as conducted at present, giving our students a real appreciation of linear relationships as a thing existent outside of books and equations? I seriously doubt it. We have been so busy with the equations of perpendicular bisectors and of pencils of lines, that we have not found time to dwell on the intrinsic meaning, and to enforce a feeling for actual quantities that may (or may not) obey laws precisely like the equations we so unmercifully manipulate. For the mass of students, I contend that it is far more important that they comprehend that a good gas meter shows readings that are in linear relation to the amount of gas consumed, and conversely that any such meter is a good meter, than it is to know how to find the distance from a given point to a given line by analytic methods.

Time will not permit more illustration. I am not especially interested in gas meters; but you may reconstruct the entire skeleton, as Agassiz could that of a prehistoric fish, from this one rib of my mental anatomy. Another would be a like comparison between the hyperbolic character of such inverse variation as that between the pressure and the volume of a gas, as contrasted with the manipulative process of rotating that same hyperbola through forty-five degrees to reduce it to a traditional form that is more complicated!

I should at least mention the calculus, though I realize that there are limits to your patience. The calculus is par excellence the study of the manner in which functions vary. It deals primarily with the rates of such changes and with the determination of quantities whose rates of change are given. My insistence would be mainly that the student should realize both this fact itself, and the meaning of it. Too often we submerge it in a maze of technique of mere manipulation of algebraic and trigonometric symbols. True command of the calculus is not told off in the number of forms one can differentiate and integrate; it is determined by the student's grasp of the reality about the rates of change of functions, and their realization that both the differential and the integral processes occur in all forms of quantitative relationships. If they do not know this to the extent of instantly thinking of it whenever relationships between quantities present themselves, then their knowledge of the calculus is imperfect, even if they have memorized, as I did when young, all the reduction formulas for binomials differentials and for trigonometric forms. I have forgotten them, but my knowledge of the calculus is on a surer foundation. What shall it profit a student though he know all the book, but have lost its soul? Aye, and it may profit him, for he may pass the course!

7. *Functional Character of Science and of Modern Life.* I have tried to point out how the idea of functional relationships pervades all of mathematics, or should pervade it, from the earliest courses in secondary schools to and beyond the calculus. Much of this has been lost in our traditional courses. Many of our students have gone from us unstirred by the vision of the relationships that exists in their own lives and in the world about them. Many of our teachers seem strangely negligent of this whole matter. Most of the critics of mathematics as a school subject seem totally unaware of its existence. Reform, and reform in this direction, is a paramount necessity for our courses if they are to survive as great school courses. And let me add that I shall raise no hand in defence of those courses that continue to neglect this central theme of all mathematics.

Functional relationships are not only the central theme of real mathematics; they constitute also its mightiest application in science and in the life of the modern world. When any modern science reaches the point in its groping for truth where measurement is possible, as most of them now have, almost the first serious question is of the relationships between associated quantities. First graphically, and then much later, when empiricism has exhausted its powers, also analytically, we seek in modern science to express as best we may the observed facts of nature, in the form of relationships between quantities. These are the laws of science. Is there a quantitative science in which this is not true? The outstanding problems of modern science, in things quantitative, are problems of functional dependence, in its purest form.

The problems of modern life are perhaps less strongly associated with functional dependence; but the associations are far more real and far more definite than has been expressed by any one. I have ventured to mention instances: life insurance, depreciation, compound interest on money. But I may add that wherever science touches modern life in quantitative aspects, the process is bound to be functional in character. For in what else would the business man be interested if not in the relationships between the quantities of the things of which the scientist is to tell him? Modern life is complex through our very knowledge of many quantitative relations. From the woman who deals with balanced rations to the man who builds a bridge, or the man who values a public service corporation, accurate habits of functional thinking are now demanded and secured. I will admit, perhaps, that we have partly failed in our mission to educate our students toward easy control of such things; but I will not admit that we cannot so educate them, nor that these problems do not exist in great number and in endless variey in the modern world.

Thus far I have spoken chiefly of what I may call the selfish or personal benefits that follow from good habits of functional thinking. I have mentioned, however, many whose broader import is of public character. Thus the control of public service corporations and of life insurance companies will either be viciously bad or it will be based upon good functional thinking, for valuation of public utilities and computation of just insurance are both fundamentally mathematical and functional in character. There are other public questions of similar nature. Control of railroad and express rates are now public rather than private questions. Taxation is emphatically public, and it is highly functional in character when it touches incomes and excess profits and unearned increments. Bond issues and redemptions, amortization, the tariff, pensions, representation, and many other phases of the public business are decidedly mathematical and decidedly functional in character.

Nor will it do to say that these questions may be solved by a few experts. Fortunately—or will you say unfortunately?—we live in democracies. In an autocracy, one outstanding advantage—or is it an advantage?—is that all such public questions may be decided by the few without consultation of the many. I need not argue to you of the abuses to which that leads. Democracy is indeed worth fighting for, and we have just won a mighty fight for it against autocracy. But democracy is postulated upon widespread understanding of public questions—upon general education of the masses as well as of the classes in all that affects the nation. Legislation must be had. Shall we have wise legislation based on good habits of functional thinking? Or is the legislation to be swayed by mere guesswork and emotion—perhaps by prejudice? We must have elections. Some of these elections will turn on questions of control of public utilities, on bond issues, on pensions, on the tariff. Are the voters to have real opinions? Or are they to be swayed by passion and prejudice, or by the influence of interested parties?

We have heard a great deal in educational discussions of this question of education for citizenship. Many a subject in our curricula bases its whole claim upon the value of the knowledge of that subject as a preparation for proper views on public questions. Thus history is not so much of selfish benefit to the individual; it is of paramount importance to the State. For voters ignorant of the historical development of our institutions, and of the trials and failures of other institutions, would be in grave danger of making fundamental mistakes. Sociology, economics, public law, and other dignified and important subjects derive at least a large share of their value from their connection with public affairs, and with the necessities of training for good citizenship.

This I will not deny, but rather affirm. Yet I may surely add that the most vital questions before the public today are those of a quantitative nature, involving relationships between quantities of no mean order. To wisely vote upon these questions, even to hold a respectable opinion regarding them, must presuppose at least some degree of a habit of functional thinking. Indeed, in our present day problems regarding bond issues, regarding pensions, regarding life insurance, regarding income taxation, regarding control of public utilities, regarding all valuations, history tells us little because our problems are either newly formed or they exist in such unprecedented size that previous experience is not a guide. Even more than history or sociology, mathematics is the key, for these problems are mathematical and functional in character. We can expect little from the great mass of the people on such problems. The danger is that this lack of comprehension of quantitative relationships may lead to disastrous decisions under some wild leadership. Is this unthinkable? Have we not seen as wild leadership and as distorted decision? We need at least a small degree of ability to think about quantities on the part of all citizens. We need leaders, politicians, statesmen, who can think very clearly on such questions, so that they may guide public opinion aright. To train such leaders is a function of the college. I feel deeply that it is of vital moment to democracy to increase as much as possible the widespread appreciation of quantitative relationships, and habits of thought concerning them. That very remarkable man, H. G. Wells, in his *Mankind in the Making,* has said: ''The new mathematics is a sort of supplement to language, affording a means of thought about form and quantity, and a means of expression, more exact, compact, and ready, than ordinary language. The great body of physical science, a great deal of the essential facts of financial science, and endless social and political problems, are only accessible and only thinkable to those who have had a sound training in mathematical analysis. The time may not be very remote when it will be understood that for complete initiation as an efficient citizen of the new great complex world-wide states that are now developing, it is necessary to be able to compute, to think in averages and in maxima and minima, as it is now understood to be necessary to be able to read and to write.''

This whole paper might well be said to be an introduction to and a preparation for this very strong statement, which is indeed so strong as to be well-nigh incomprehensible without some preparation. With it I would rest my case. But let me warn those of you who are teachers of mathematics that Wells does not mean our traditional formalized courses when he speaks of the ''new'' mathematics. He means a mathematics reborn, revitalized, conscious of its soul and of its destiny. His praise is not for smug content with what is commonly thought of when mathematics is mentioned. If we base the case of mathematics on such statements as his, or on those of this paper, or on those of the National Committee which I quoted in the beginning, we must remake our courses, we must abandon many of our traditions, we must shape our work so that it will indeep help to give our students real and correct habits of functional thinking in all those quantitative situations which they will meet in their private lives and in their public activities.